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The calculation of the energy losses in a sliding elastic contact with friction and wear (the plane problem)☆

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Abstract

The plane problem of the sliding contact of a punch with an elastic foundation when there is friction and wear is considered. Assuming the existence of a steady solution in a moving system of coordinates, relations are derived between the sliding velocity, the wear, the contact stresses and the displacements for an arbitrary dependence of the wear rate on the contact pressure. Taking into account the presence of a deformation component of the friction force, an equation is written for the balance of the mechanical energy for the punch - elastic base system considered. It is shown that the equality of the work of the external force in displacing the punch to the losses due to friction and the change in the shape of the foundation due to wear is satisfied when the work done by the contact stresses on the increments of the boundary displacements is equal to zero, and the frictional losses must be determined taking into account the non-uniformity of the distributions of the shear contact stresses and the sliding velocity in the contact area. Two special cases of the foundation in the form of a wide and narrow strip are considered, for which the total coefficient of friction is calculated, taking into account the deformation component of the friction force.

The resistance to the relative motion of bodies in contact due to their irreversible deformation was apparently calculated for the first time in Ref. 1, when analysing a method of determining the hardness to scratching. Later, the asymmetry of the contact pressure profile was regarded as the reason for the occurrence of a rolling friction force.[2,3] A fairly complete review of research on the deformation component of the friction force can be found in Ref. 4.

According to existing results,[5,6] when elastic bodies are in contact under conditions of sliding and wear, the contact pressure profile is asymmetrical, which also leads to the occurrence of a deformation component of the friction force. In this connection, the problem arises of satisfying the balance of mechanical energy, under the condition that it is not dissipated in the elastic body.

1. Formulation of the problem and fundamental relations

We will consider the two-dimensional problem for a deformable foundation with a rectilinear boundary Γ, on which an absolutely rigid smooth punch slides (see the Fig. 1). We will connect with the foundation a system of coordinates $\Omega\xi\eta$, the ξ axis of which is parallel to the undeformed boundary Γ and directed along the motion of the punch. The external load on the punch is specified by a shear component T and a normal component P, directed along the ξ axis and

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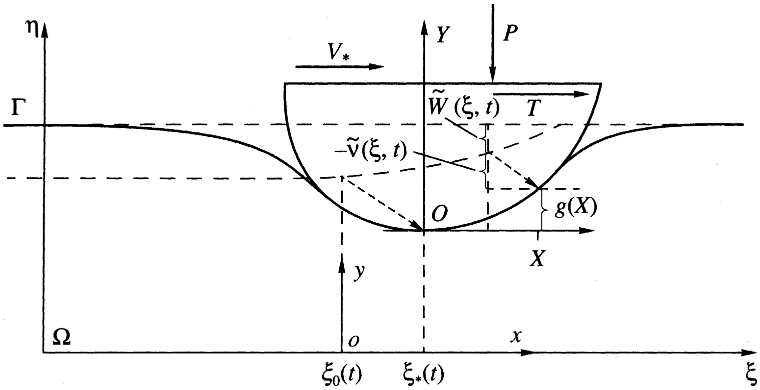

Fig. 1.

opposite to the η axis, respectively (see the Fig. 1). It is assumed that an external moment is applied to the punch, which gives it a plane-parallel displacement without rotation. The interaction of the punch and the foundation is accompanied by friction, which leads to wear of the foundation.

We will use the Lagrangian description of the deformations[7] and we will denote by $\tilde{u}(\xi, \eta)$ and $\tilde{v}(\xi, \eta)$ the displacements of the boundary Γ, and we will denote the contact stresses by

$$\tilde{q}_1(\xi, t) = \tau_{\xi\eta}(\xi, \eta, t)|_{\xi, \eta \in \Gamma}, \quad \tilde{q}_2(\xi, t) = -\sigma_\eta(\xi, \eta, t)|_{\xi, \eta \in \Gamma}$$

The area of contact of the punch with the foundation will be represented by the segment $[\alpha_1(t), \alpha_2(t)]$. The argument t here denotes the dependence of quantities on time. It is assumed that the rate of linear wear $\tilde{W}$ of the foundation is related to the contact pressure $\tilde{q}_2$ and the velocity $\tilde{V}$ of relative sliding of the boundaries of the punch and the foundation by the wear law

$$\partial\tilde{W}(\xi, t)/\partial t = F(\tilde{q}_2(\xi, t), \tilde{V}(\xi, t)) \tag{1.1}$$

in which $F(\tilde{q}_2, \tilde{V})$ is a known function.

We will take a certain point O on the boundary of the punch and connect with it a moving system of coordinates OXY, in which the X and Y axes are directed along the ξ and η axes respectively (see the Fig. 1). We will define the shape of the punch by the equation

$$Y = g(X) \tag{1.2}$$

where $g(X)$ is a given continuous function, and $g(0) = 0$. We will denote by $\xi_0(t)$ the Lagrangian coordinate ξ of the point of the boundary Γ coinciding with the point O at the instant t, and we will assume that $\xi_0(t) = V_0 t$, $0 < V_0 = \text{const}$. In the deformed state, this point has the coordinate

$$\xi_*(t) = \xi_0(t) + \tilde{u}(\xi_0(t), t) = V_0 t + \tilde{u}(\xi_0(t), t)$$

Here, the following expression holds for the velocity of motion of the punch in the system of coordinates Ωξη

$$V_*(t) \equiv d\xi_*(t)/dt = V_0 + d\tilde{u}(\xi_0(t), t)/dt \tag{1.3}$$

We will introduce one more moving system of coordinates oxy, the origin of which coincides with the point $\xi_0(t)$ of the coordinate x^ξ (see the Fig. 1). The coordinates x and ξ at each instant of time are related by the equality

$$x = \xi - \xi_0(t), \quad \xi_0(t) = V_0 t \tag{1.4}$$

In what follows we will consider the case of the steady motion of the medium of the foundation in the system oxy.[2,8] The corresponding boundary displacements in this system, taking (1.4) into account, have the form

$$u(\xi - \xi_0(t)) = \tilde{u}(\xi, t) - \tilde{u}(\xi_0(t), t), \quad \upsilon(\xi - \xi_0(t)) = \tilde{\upsilon}(\xi, t) - \tilde{\upsilon}(\xi_0(t), t) \tag{1.5}$$

Also, for the system oxy we will introduce contact stresses q_1 and q_2, the dimensions a and b of the contact area, the wear W and the velocity V of relative sliding of the boundaries of the punch and the foundation

$$q_k(x) = \tilde{q}_k(x+\xi_0(t), t), \quad k = 1, 2; \quad -a = \alpha_1(t)-\xi_0(t), \quad b = \alpha_2(t)-\xi_0(t)$$
$$W(x) = \tilde{W}(x+\xi_0(t), t), \quad V(x) = \tilde{V}(x+\xi_0(t), t) \tag{1.6}$$

Relations (1.5) enable us to obtain the velocity V. In fact, by definition

$$\tilde{V}(\xi, t) = V_*(t) - \partial\tilde{u}(\xi, t)/\partial t$$

If we represent the derivative in this equation using Eq. (1.5) in the form

$$\partial\tilde{u}(\xi, t)/\partial t = d\tilde{u}(\xi_0(t), t)/dt - V_0 u'(\xi-\xi_0(t))$$

and take expression (1.3) into account, it turns out that $\tilde{V}(\xi, t) = V_0 + V_0 u'(\xi - \xi_0(t))$, or

$$V(x) = V_0[1+u'(x)] \tag{1.7}$$

A similar result was obtained previously in Ref. 8 when considering the deformation of a Winkler foundation by a rolling cylinder.

We now take a certain point on the boundary Γ with coordinate $\xi \in [\alpha_1(t), \alpha_2(t)]$. Suppose the coordinate of this point in the deformed state of the foundation lies at a distance X from the point O, so that, taking relations (1.4) and (1.5) into account, we have

$$X \equiv [\xi + \tilde{u}(\xi, t)] - [\xi_0(t) + \tilde{u}(\xi_0(t), t)] = x + u(x) \tag{1.8}$$

The condition for a punch, the shape of which is described by Eq. (1.2), to be in contact with the foundation is expressed by the equality (see the Fig. 1)

$$\tilde{W}(\xi, t) - \tilde{\upsilon}(\xi, t) + g(X) = \tilde{W}(\xi_0(t), t) - \tilde{\upsilon}(\xi_0(t), t), \quad \xi \in [\alpha_1(t), \alpha_2(t)]$$

which, taking relations (1.5), (1.6) and (1.8) into account, can be given the form

$$\upsilon(x) - W(x) = g(x+u(x)) - W(0), \quad x \in [-a, b] \tag{1.9}$$

Note that the contact condition (1.9) corresponds to the refined formulation of the contact problem, which takes into account the tangential boundary displacement in the boundary condition for a normal displacement.[9,10]

The wear W occurring in condition (1.9) is related to the contact pressure q_2 by virtue of wear law (1.1). In fact, for fixed ξ, by relations (1.4) and (1.6)

$$q_k(x) = \tilde{q}_k\left(\xi, \frac{\xi-x}{V_0}\right), \quad k = 1, 2; \quad W(x) = \tilde{W}\left(\xi, \frac{\xi-x}{V_0}\right), \quad V(x) = \tilde{V}\left(\xi, \frac{\xi-x}{V_0}\right)$$

Hence, taking Eq. (1.1) into account, we can write the chain of equalities

$$W'(x) = \frac{d}{dx}\tilde{W}\left(\xi, \frac{\xi-x}{V_0}\right) = -\frac{1}{V_0}\frac{\partial\tilde{W}(\xi, t)}{\partial t}\bigg|_{t=t_x} = -\frac{1}{V_0}F(\tilde{q}_2(\xi, t), \tilde{V}(\xi, t))\big|_{t=t_x}$$

where $t_x = (\xi - x)/V_0$, and as a result we obtain

$$W'(x) = -\frac{1}{V_0}F(q_2(x), V(x)) \tag{1.10}$$

Below we will use the linear theory of deformations,[7] within the framework of which it is assumed that the quantities $|u'(x)|$, $|\upsilon'(x)|$ are of the order of smallness $\varepsilon \ll 1$, and quantities of higher orders of smallness are omitted. The derivative $W'(x)$ will be assumed to be of the same order of smallness. These assumptions enable us, if necessary, to drop quantities of higher order of smallness than the other quantities in the equations. In particular, taking into account the fact that

$$x + u(x) = x(1+u'(\bar{x})) = x(1+O(\varepsilon))$$

where $\bar{x}$ lies between 0 and x, we can represent the contact condition (1.9) in the simplified form

$$\upsilon(x) - W(x) = g(x) - W(0) \quad x \in [-a, b] \tag{1.11}$$

Note that the use of the contact condition (1.11) instead of (1.9) leads to corrections $O(\varepsilon)$ to the solution of the corresponding contact problem.[10]

The relations obtained above do not assume any particular behaviour of the medium of the foundation. However, the following description will be concerned with an elastic foundation.

2. Energy relations

For small values of $|g'(x)|$, the forces acting on the punch must satisfy the following equilibrium conditions[2,5]

$$T = Q_1 + \psi, \quad P = Q_2 - \varphi \tag{2.1}$$

where

$$Q_k = \int_{-a}^{b} q_k(x)dx, \quad k = 1, 2; \quad \begin{Bmatrix} \varphi \\ \psi \end{Bmatrix} = \int_{-a}^{b} \begin{Bmatrix} q_1(x) \\ q_2(x) \end{Bmatrix} g'(x)dx \tag{2.2}$$

Q_1 is the total friction force, and ψ is the deformation (mechanical) component of the friction force, related to the asymmetry of the contact- pressure distribution.[2,4,5] In the case of an elastic foundation, this asymmetry is due to friction on the contact and wear of the foundation.[6] Further analysis will involve considering the deformation component of the friction force, and hence the quantity ψ is retained in Eq. (2.1) despite the fact that it has a higher order of smallness compared with Q_1.

Taking into account the assumption that the motion of the medium of the foundation in the moving system of coordinates oxy is independent of time, we will assume that the displacement $\tilde{u}(\xi_0(t), t)$ of a point coinciding with the point O of the punch boundary also does not change with time, i.e.

$$\frac{d}{dt}\tilde{u}(\xi_0(t), t) = 0 \tag{2.3}$$

so that, according to expression (1.3), $V_* = V_0$. Below, Eq. (2.3) will be confirmed for a foundation in the form of an elastic strip.

For a power M_T of the external shear force T, taking into account the equality $V_* = V_0$ and the first relation of (2.1), we have

$$M_T = M_1 + M_\psi; \quad M_1 = Q_1 V_0, \quad M_\psi = \psi V_0 \tag{2.4}$$

where M_1 is the power of the total friction force and M_ψ is the power of the deformation component of the friction force. Expression (2.4) means that, if the losses due to friction are defined in terms of the work of the total friction force, as is done in the case of the friction of solid bodies, the work of the external force T is only partially expended in covering friction losses. In relation to the problem of where the remainder M_ψ of the power M_T is expended, whereas there is no dissipation of energy in the elastic foundation, we will determine the losses due to friction taking into account the non-uniformity of the distributions of the shear stress q_1 and the velocity V of relative sliding in the contact area. Moreover, by analogy with the consideration of the contribution of the friction force of plastic deformation to the deformation component,[4] we will take into account the losses due to the irreversible change in the shape of the boundary of the foundation due to its wear.

Using relation (1.7), we will represent the rate of loss of energy due to friction in the form

$$\dot{D}_1 \equiv \int_{-a}^{b} q_1(x)V(x)dx = M_1 + V_0 I, \quad I \equiv \int_{-a}^{b} q_1(x)u'(x)dx \tag{2.5}$$

The rate of loss of energy due to the change in the shape of the foundation due to its wear is given by the equation

$$\dot{D}_W \equiv \int_{\alpha_1(t)}^{\alpha_2(t)} \tilde{q}_2(\xi, t)\frac{\partial \tilde{W}(\xi, t)}{\partial t}d\xi \tag{2.6}$$

since $\tilde{q}_2 d\tilde{W}$ is the elementary work for a displacement $d\tilde{W}$ of the boundary of the foundation. If, in relation (2.6), we change from the variable ψ to x, using equality (1.4), by taking relations (1.1), (1.10) and (1.11) and the definitions of the quantities ψ and M_ψ into account, we can write

$$\dot{D}_W = -V_0\int_{-a}^{b} q_2(x)W'(x)dx = M_\psi - V_0 J, \quad J \equiv \int_{-a}^{b} q_2(x)v'(x)dx \tag{2.7}$$

By the law of conservation of energy, the work of the external force T is dissipated in friction and a change in the shape of the foundation due to its wear, i.e. the following relation must be satisfied

$$M_T = \dot{D}_1 + \dot{D}_W \tag{2.8}$$

Expressions (2.4), (2.5) and (2.7) enable us to reduce relation (2.8) to the equality

$$I \equiv \int_{-a}^{b} q_1(x)u'(x)dx = \int_{-a}^{b} q_2(x)v'(x)dx \equiv J \tag{2.9}$$

the validity of which must be established using equations describing the deformation of the foundation. Below, we will do this for the case of a foundation in the form of an elastic strip.

Remarks.

1°. The equality $I - J = 0$, equivalent to (2.9), can be interpreted as the equality to zero of the total work of the stresses $\tau_{xy} = q_1$, $\sigma_y = -q_2$, distributed over the contact area, for corresponding increments $du = u'dx$, $dv = v'dx$ of the boundary displacements. A similar construction is used when formulating the virtual principle work.[7]

2°. Equality (2.9) enables us to draw the physically natural conclusion that the deformation component ψ of the friction force for an arbitrary shape $g(X)$ of the punch when there is no friction and wear on the contact is equal to zero. In fact, when $q_1(x) \equiv 0$, $W(x) \equiv 0$, Eq. (2.9) and the contact condition (1.11), together with the definition (2.2), give $I = J = 0$ and $\psi = J$, i.e. $\psi = 0$.

3. The case of an elastic strip

Suppose the foundation is an elastic strip, the lower boundary of which is attached to an absolutely rigid substrate. Then, in the quasi-static approximation (see Ref. 11, Chapter 2)

$$\begin{aligned} m\tilde{u}(\xi, t) &= \int_{\alpha_1(t)}^{\alpha_2(t)} \tilde{q}_1(\rho, t)k_{11}\left(\frac{\rho-\xi}{h}\right)d\rho + \chi\int_{\alpha_1(t)}^{\alpha_2(t)} \tilde{q}_2(\rho, t)k_{12}\left(\frac{\rho-\xi}{h}\right)d\rho \\ m\tilde{v}(\xi, t) &= \chi\int_{\alpha_1(t)}^{\alpha_2(t)} \tilde{q}_1(\rho, t)k_{12}\left(\frac{\rho-\xi}{h}\right)d\rho - \int_{\alpha_1(t)}^{\alpha_2(t)} \tilde{q}_2(\rho, t)k_{22}\left(\frac{\rho-\xi}{h}\right)d\rho \end{aligned} \tag{3.1}$$

where $m = \pi E/[2(1-\nu^2)]$, $\chi = (1-2\nu)/[2(1-\nu)]$

где $m = \pi E/[2(1-\nu^2)]$, $\chi = (1-2\nu)/[2(1-\nu)]$

$$\begin{Bmatrix} k_{jj}(z) \\ k_{12}(z) \end{Bmatrix} = \int_0^\infty \begin{Bmatrix} K_{jj}(X)\cos zX \\ K_{12}(X)\sin zX \end{Bmatrix} \frac{dX}{X}$$

$$K_j(X) = \frac{2\kappa \operatorname{sh} 2X - (-1)^j 4X}{D(X)}, \quad K_{12}(X) = \frac{2\kappa(\cos 2X - 1) - 8(\kappa - 1)^{-1}X^2}{D(X)} \tag{3.2}$$

$$D(X) = 2\kappa \operatorname{ch} 2X + 4X^2 + 1 + \kappa^2, \quad \kappa = 3 - 4\nu$$

E and ν are Young's modulus and Poisson's ratio, and h is the width of the strip. Here and henceforth $i, j = 1, 2$.

Changing in Eq. (3.1) from the variable ξ to x, according to Eq. (1.4) and taking into account Eqs. (1.5) and (1.6), we can write the following relations

$$m[u(x) + \tilde{u}(\xi_0(t), t)] = \int_{-a}^{b} q_1(s) k_{11}\left(\frac{s-x}{h}\right) ds + \chi \int_{-a}^{b} q_2(s) k_{12}\left(\frac{s-x}{h}\right) ds$$

$$m[v(x) + \tilde{v}(\xi_0(t), t)] = \chi \int_{-a}^{b} q_1(s) k_{12}\left(\frac{s-x}{h}\right) ds - \int_{-a}^{b} q_2(s) k_{22}\left(\frac{s-x}{h}\right) ds \tag{3.3}$$

the form of which immediately enables us to establish the validity of assumption (2.3) made above. For a further analysis of these relations, denoting by $H\,[-a, b]$ the class of functions which satisfy the Hölder condition in the segment $[-a, b]$,[12] we will assume that

$$q_k(x) \in H[-a, b], \quad k = 1, 2 \tag{3.4}$$

In order to check equality (2.9) we will obtain from relations (3.3) expressions for the derivatives $u'(x)$ and $v'(x)$. Direct differentiation with respect to x of the right-hand sides of equalities (3.3) does not give the required result, since, according to relations (3.2), the derivatives of the kernels $k_{ij}(z)$ are represented by diverging integrals, which do not enable us to introduce the operation of differentiation under the integral signs in relations (3.3).[13] We will therefore convert these kernels, excluding the singularities. We have

$$K_j(X) = L_j(X) + \operatorname{th} X, \quad K_{12}(X) = L_{12}(X) + 1$$

$$L_j(X) = -\frac{(4X^2 + 1 + \kappa^2)\operatorname{th} X + (-1)^j 4X}{D(X)}, \quad L_{12}(X) = -\frac{4(1 + 2(\kappa - 1)^{-1})X^2 + (\kappa + 1)^2}{D(X)}$$

We will define the kernels

$$l_{jj}(z) = \int_0^\infty L_j(X) \frac{\cos zX}{X} dX - \ln\left|\frac{1}{z}\operatorname{th}\frac{\pi}{4}z\right|, \quad l_{12}(z) = \int_0^\infty L_{12}(X) \frac{\sin zX}{X} dX \tag{3.5}$$

so that

$$k_{jj}(z) = l_{jj}(z) - \ln|z|, \quad k_{12}(z) = l_{12}(z) + \frac{\pi}{2}\operatorname{sign} z \tag{3.6}$$

Using expressions (3.5) it can be shown that the kernels $l_{ij}(z)$ enable us to carry out differentiation under the sign of the corresponding integrals. Moreover, with condition (3.4), using the Poincaré - Bertrand formula,[12] we can establish the relation (everywhere henceforth the integration is carried out over the segment $[-a, b]$)

$$\frac{d}{dx}\int q_k(s) \ln\left|\frac{s-x}{h}\right| ds = -(\mathcal{H} q_k)(x), \quad (\mathcal{H} q_k)(x) = \int \frac{q_k(s)}{s-x} ds, \quad x \in (-a, b)$$

All this enables us, by replacing $k_{ij}(z)$ by $l_{ij}(z)$ in relations (3.3), according to equalities (3.6), to differentiate the right-hand sides of the relations obtained with respect to x with the introduction of corresponding operations under the integral signs[13] and, as a result, we arrive at the following expressions

$$\begin{aligned} mu'(x) &= -\pi\chi q_2(x) + (\mathcal{H}q_1)(x) - \frac{1}{h}\int q_1(s)l'_{11}\left(\frac{s-x}{h}\right)ds - \frac{\chi}{h}\int q_2(s)l'_{12}\left(\frac{s-x}{h}\right)ds \equiv \\ &\equiv \int q_1(s)n_{11}\left(\frac{s-x}{h}\right)ds + \int q_2(s)n_{12}\left(\frac{s-x}{h}\right)ds \\ mv'(x) &= -\pi\chi q_1(x) - (\mathcal{H}q_2)(x) - \frac{\chi}{h}\int q_1(s)l'_{12}\left(\frac{s-x}{h}\right)ds + \frac{1}{h}\int q_2(s)l'_{22}\left(\frac{s-x}{h}\right)ds \equiv \\ &\equiv \int q_1(s)n_{12}\left(\frac{s-x}{h}\right)ds - \int q_2(s)n_{22}\left(\frac{s-x}{h}\right)ds \end{aligned} \tag{3.7}$$

in which, with the formal use of the idea of the δ-function, we have introduced the notation

$$n_{jj}(z) = \frac{1}{h}[z^{-1} - l'_{jj}(z)], \quad n_{12}(z) = -\frac{\chi}{h}[\pi\delta(z) + l'_{12}(z)] \tag{3.8}$$

Note that when $h \to \infty$ relations (3.7), convert, as they should do, into the well-known relations for an elastic half-plane.[14]

We will now consider equality (2.9) and replace the derivatives $u'(x)$ and $v'(x)$ in them using formulae (3.7). The result can be represented in the form

$$S_{11}^{(11)} + S_{12}^{(12)} = S_{21}^{(12)} - S_{22}^{(22)}; \quad \begin{Bmatrix} S_{kl}^{(jj)} \\ S_{kl}^{(12)} \end{Bmatrix} \equiv \int q_k(x)\left[\int q_l(s)n_{\begin{Bmatrix} jj \\ 12 \end{Bmatrix}}\left(\frac{s-x}{h}\right)ds\right]dx, \ l = 1, 2 \tag{3.9}$$

Taking expressions (3.8) for the kernels $n_{jj}(z)$ into account, in which the functions $l'_{jj}(z)$ are continuous and odd, using the Poincaré - Bertrand formula[12] we can change the order of integration in the definition of $S_{kk}^{(jj)}$ and establish that $S_{kk}^{(jj)} = -S_{kk}^{(jj)}$, i.e. $S_{kk}^{(jj)} = 0$. Carrying out similar action with respect to the quantity $S_{12}^{(12)}$, taking into account the evenness of the continuous function $l'_{12}(z)$ in Eq. (3.8) we obtain $S_{12}^{(12)} = S_{21}^{(12)}$. The last two equations indicate the correctness of equality (3.9), and, consequently, Eq. (2.9) also. According to the discussion in Section 2, this means that for a foundation in the form of an elastic strip the power balance (2.8), which expresses the law of conservation of energy, holds.

We will now consider two special cases: a wide and marrow strip of width h. To be specific we will assume that the contact stresses q_1 and q_2 for the chosen direction of motion of the punch (see the Fig. 1) are related by the Amonton - Coulomb law[4]

$$q_1(x) = \mu q_2(x) + \tau_0 \tag{3.10}$$

in which μ is the coefficient of friction and τ_0 is the adhesive (molecular) component of the friction. In this case the equilibrium conditions (2.1) take the form

$$T = \mu Q_2 + T_0 + \psi, \quad P = Q_2 - \mu\psi - \Delta_g\tau_0; \quad T_0 = \tau_0(a+b); \quad \Delta_g = g(b) - g(-a) \tag{3.11}$$

(T_0 is the total adhesive component of the friction force).

Introducing the total coefficient of friction μ_*, which takes into account the deformation components φ and ψ of the external forces T and P, so that

$$T = \mu_* P + T_0 \tag{3.12}$$

after substituting expressions (3.11) into Eq. (3.12) we obtain

$$\mu_* = \mu + [(1+\mu^2)\psi + \mu\Delta_g\tau_0]/P \tag{3.13}$$

4. A wide strip

Suppose the following asymptotic form $(a+b)/h \to 0$ holds for the strip. In this case relations (3.7) take the form[15]

$$mu'(x) = -\pi\chi q_2(x) + (\mathcal{H}q_1)(x) - \chi\frac{c_{12}}{h}Q_2,\ mv'(x) = -\pi\chi q_1(x) - (\mathcal{H}q_2)(x) - \chi\frac{c_{12}}{h}Q_1 \quad (4.1)$$

where the quantity c_{12} depends only on Poisson's ratio ν. The punch will be assumed to be parabolic: $g(x) = x^2/(2R)$, where R is the radius of curvature. We will assume the wear law (1.1) to be linear: $F(q_2, V) = k_w q_2 V$, so that, by relations (1.7) and (1.10), when $|u(x)| = O(\varepsilon) \ll 1$ we obtain

$$W'(x) = -k_W q_2(x) \quad (4.2)$$

To solve the corresponding wear-contact problem we differentiate the contact condition (1.11) with respect to x and eliminate the derivatives $v'(x)$ and $W'(x)$ from it using Eqs (4.1) and (4.2). Then expressing the stress q_1 in terms of q_2, using the friction law (3.10), we obtain the following equation for the contact pressure

$$\gamma q_2(x) + \frac{1}{m}(\mathcal{H}q_2)(x) = \Phi(x), \quad x \in [-a, b] \quad (4.3)$$

where

$$\gamma = \frac{\pi\chi}{m}\mu - k_W, \quad \Phi(x) = -\frac{x}{R} - \Phi_0, \quad \Phi_0 = \frac{\pi\chi}{m}\tau_0 + \frac{\chi c_{12}}{mh}Q_1$$

The solution of Eq. (4.3) which satisfies condition (3.4) has the form[6,12]

$$q_2(x) = \left[\gamma R\sqrt{1 + \left(\frac{\pi}{\gamma m}\right)^2}\right]^{-1}(a+x)^{1/2-\theta}(b-x)^{1/2+\theta}, \quad x \in [-a, b] \quad (4.4)$$

Here the following equality must be satisfied

$$\frac{a-b}{2} = \theta(a+b) + R\Phi_0; \quad \theta = \frac{1}{\pi}\operatorname{arctg}\frac{m\gamma}{\pi} \in \left(-\frac{1}{2}, \frac{1}{2}\right) \quad (4.5)$$

Integration of expression (4.4) over the contact area, taking into account definition (2.2) of the quantity Q_2 gives one more equality

$$(a+b)^2 = \frac{8RQ_2}{(1-4\theta^2)m} \quad (4.6)$$

Eqs. (4.5) and (4.6) serve to find the unknown dimensions a and b of the contact area.

Solution (4.4) enables us to obtain the deformation component ψ of the friction force, defined by the last formula of (2.2). After evaluating the corresponding integral, bearing equality (4.4) in mind, we have

$$\psi = -\frac{Q_2}{R}\left[\frac{4}{3}\theta(a+b) + R\Phi_0\right] \quad (4.7)$$

Moreover, it follows from Eq. (4.5) that

$$\Delta_g \equiv \frac{1}{2R}(b^2 - a^2) = -\frac{a+b}{R}[\theta(a+b) + R\Phi_0] \quad (4.8)$$

The sum $a+b$ in expressions (4.7) and (4.8) depends on the unknown quantity Q_2, by virtue of relation (4.6), and hence the substitution of these expressions into the second equality of (3.11) enables us to change to an algebraic equation defining Q_2 in terms of the known load P. Using the solution of this equation we can establish a relation between the quantities ψ, Δ_g and P on the basis of equalities (4.7) and (4.8), after which expression (3.13) enables us to determine how the total coefficient of friction μ_* depends on the load P.

In the special case when the strip degenerates into a half-plane ($h^{-1}=0$), there is no adhesive component of the friction ($\tau_0=0$) and $\mu \ll 1$

$$\mu_* = \mu - \frac{8}{3}\sqrt{\frac{2\theta^2}{(1-4\theta^2)}\frac{P}{mR}} \tag{4.9}$$

As the expression obtained shows, the value of μ_* turns out to be less than the coefficient of friction μ. This is due to the fact that, in the case considered, according to equality (4.5), the difference *a-b* is positive, i.e. the contact pressure profile is shifted in a direction opposite to the direction of motion of the punch.[6]

5. A narrow strip

Suppose the asymptotic form $(a+b)/h \to \infty$ is satisfied for the strip. In this case integral relations (3.1) degenerate into algebraic relations, corresponding to the Winkler model.[11,16] For a moving system of coordinates oxy these relations have the form

$$u(x) = \alpha[q_1(x) - q_1(0)], \quad \upsilon(x) = -\beta[q_2(x) - q_2(0)] \tag{5.1}$$

where

$$\alpha = 2(1+\nu)hE^{-1}, \quad \beta = (1-2\nu)(1+\nu)(1-\nu)^{-1}hE^{-1}$$

Note that relations (5.1) are not asymptotically accurate – they break down in the neighbourhoods $O(h)$ at the ends of the contact area,[11,16] in view of which we must separately verify equality (2.9) for relations (5.1). We will carry out this check assuming that the contact pressure has zero values at the ends of the contact area:

$$q_2(-a) = q_2(b) = 0 \tag{5.2}$$

and that a friction law of general form $q_1(x) = \Pi(q_2(x))$ holds, where $\Pi(q_2)$ is a known function. We substitute the first relation of (5.1) into expression (2.9) for I and we integrate the integral obtained by parts, taking into account the last three equalities. As a result we can write the equality

$$I \equiv \alpha\int q_1(x)q_1'(x)dx = \alpha q_1^2(x)\big|_{-a}^{b} - \alpha\int q_1(x)q_1'(x)dx = -I$$

which denotes that $I=0$. We can similarly establish the equality $J=0$, and, consequently, also the required equality (2.9).

We will obtain a solution of the wear-contact problem for a narrow strip, assuming the wear law (1.1) to be linear: $F(q_2, V) = k_w q_2 V$. Differentiation of the contact condition (1.11) with respect to x and the use of expressions (4.2) and (5.1) for $W'(x)$ and $\upsilon(x)$ enables us to obtain the following differential equation for the contact pressure

$$q_2'(x) - \lambda q_2(x) = -\beta^{-1}g'(x), \quad x \in [-a, b]; \quad \lambda = k_W/\beta \tag{5.3}$$

The solution of Eq. (5.3) which satisfies condition (5.2) has the form

$$q_2(x) = -\beta^{-1}e^{\lambda x}\int_{-a}^{x} g'(s)e^{-\lambda s}ds, \quad x \in [-a, b] \tag{5.4}$$

Here the following equality must be satisfied

$$\int g'(x)e^{-\lambda x}dx = 0 \tag{5.5}$$

Integration of expression (5.4) over the contact area, taking into account definition (2.2) of the quantity Q_2, gives one more equality

$$g(b) - g(-a) = \beta\lambda Q_2 \tag{5.6}$$

Equalities (5.5) and (5.6) serve to obtain the unknown dimensions a and b of the contact area.

We will further assume that the punch is parabolic: $g(x) = x^2/(2R)$. Relations (5.4) and (5.6) then take the form

$$q_2(x) = \frac{1}{\beta\lambda R}f(x), \quad f(x) = x + ae^{\lambda(a+x)} + \frac{1}{\lambda}(1 - e^{\lambda(a+x)}), \quad x \in [-a, b] \tag{5.7}$$

$$f(b) = 0, \quad b^2 - a^2 = 2R\beta\lambda Q_2 \tag{5.8}$$

Substitution of the solution of (5.7) into definition (2.2) of the deformation component ψ of the friction force, taking into account the first equality of (5.8), enables us to obtain the expression

$$\psi = \frac{1}{2\beta\lambda R^2}\left[\frac{2}{3}(a^3 + b^3) - \frac{b^2 - a^2}{\lambda}\right] \tag{5.9}$$

Moreover, we have directly from the second equality of (5.8)

$$\Delta_g \equiv \frac{1}{2R}(b^2 - a^2) = \beta\lambda Q_2 \tag{5.10}$$

The dimensions a and b of the contact area, present in expressions (5.9) and (5.10), are expressed in terms of the unknown quantity Q_2, by virtue of relations (5.8), and hence by substituting these expressions into the second equality of (3.11) we can arrive at a transcendental equation, defining Q_2 in terms of the known load P. Using the solution of this equation we can establish a connection between the quantities ψ, Δ_g and P on the basis of Eqs. (5.9) and (5.10), after which, expression (3.13) enables us to determine how the total coefficient of friction μ_* depends on the load P.

In the special case when the wear is small ($\lambda(a+b) \to 0$) and there is no adhesive component of the friction ($\tau = 0$), we have

$$\mu_* = \mu + \frac{3}{5}(1 + \mu^2)k_W\left(\frac{2}{3}\frac{P^2}{\beta R}\right)^{1/3}$$

Unlike expression (4.9), here the value of μ_* turns out to be greater than the coefficient of friction μ. This is due to the fact that, in the case considered, according to the second equality of (5.8), the difference a-b is negative, i.e. the profile of the contact pressure is shifted in the direction of motion of the punch.

6. Conclusions

1. We have obtained a relation (1.7) between the sliding velocity and the tangential boundary displacement, and also relation (1.10) between the wear and the contact pressure for an arbitrary wear law.
2. We have set up Eq. (2.8) of the balance of mechanical energy, which takes into account the losses due to friction and the irreversible change in the shape of the boundary of the foundation as a result of wear.
3. We have shown that the balance of the mechanical energy is satisfied under condition (2.9) that the work of the contact stresses for increments of the boundary displacements is equal to zero, and the frictional losses must be determined taking into account the non-uniformity of the distributions of the shear contact stresses and the sliding velocity in the contact area.
4. We have shown that equality (2.9) is satisfied for a foundation in the form of an elastic strip of arbitrary width.

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